Sheaves and $\mathcal{D}$-modules in integral geometry

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ABSTRACT. Integral geometry deals with those integral transforms which associate to "functions" on a manifold X their integrals along submanifolds parameterized by another manifold Ξ. Basic problems in this context are range characterization—where systems of PDE appear—and inversion formulae. As we pointed out in a series of joint papers with Pierre Schapira, the language of sheaves and $\mathcal{D}$-modules provides both a natural framework and powerful tools for the study of such problems. In particular, it provides a general adjunction formula which is a sort of archetypical theorem in integral geometry. Focusing on range characterization, we illustrate this approach with a discussion of the Radon transform, in some of its manifold manifestations.

Introduction

According to I. M. Gelfand, integral geometry is the study of those integral transforms $\varphi(x) \mapsto \psi(\xi) = \int \varphi(x) k(x, \xi)$ which are associated to a geometric correspondence $S \subset X \times \Xi$ between two manifolds X and Ξ. In other words, given the family of subvarieties $\widehat{\xi} = \{x : (x, \xi) \in S\} \subset X$, parameterized by the manifold $\Xi \ni \xi$, one is interested in those transforms whose integral kernel k is a characteristic class of the incidence relation S. Typical examples are the various instances of the Radon transform. For the real affine Radon transform, $\widehat{\xi}$ is a p-plane in an affine space X, and $\psi(\xi) = \int_{\widehat{\xi}} \varphi(x)$ is the integral of a rapidly decreasing C^∞-function with respect to the Euclidean measure. For the complex projective Radon transform, $\widehat{\xi}$ is a p-plane in a complex projective space X, φ is a cohomology class of a holomorphic line bundle, and the integral $\int_{\widehat{\xi}}$ has to be understood as a Leray-Grothendieck residue. The conformal Radon transform may be viewed as a boundary value of the complex projective transform and, in this case, the natural function space to consider is that of hyperfunctions. We begin by recalling in the first section some classical results on range characterization for these Radon transforms, where systems of partial differential equations appear.

In the second and main part of this survey, we present a general framework for integral geometry, based on the theory of sheaves and $\mathcal{D}$-modules, which we proposed in a series of joint papers with Pierre Schapira. This approach makes apparent the common pattern underlying the various instances of the Radon transform. Let us discuss it in some detail.

It is a central idea of M. Sato to obtain hyperfunctions on a real analytic manifold M as boundary value of holomorphic functions in a complex neighborhood $X \supset M$. More generally, one may obtain generalized functions by coupling holomorphic functions with a constructible sheaf F on X. Following Kashiwara-Schapira, we consider four examples of such generalized functions, that we denote by $\mathcal{C}^\omega(F)$, $\mathcal{C}^\infty(F)$, $\mathcal{C}^{-\infty}(F)$, $\mathcal{C}^{-\omega}(F)$. If F is the constant sheaf along M, these are the sheaves of analytic functions, C^∞-functions, distributions and hyperfunctions, respectively. Let thus X and Ξ be complex manifolds. An integral transform $\int \varphi(x)k(x,\xi)$ consists of three operations. Namely, a pull-back from X to $X \times \Xi$, a product with an integral kernel, and a push-forward to Ξ. The Grothendieck formalism of operations makes sense in the categories of sheaves and $\mathcal{D}$-modules. Let us denote here for short by $\widehat{\mathcal{M}}$ the transform of a coherent $\mathcal{D}$-module $\mathcal{M}$ on X, and by $\widehat{G}$ the transform of a constructible sheaf G on Ξ. A general result in this framework asserts that, under mild hypotheses, solutions of $\mathcal{M}$ with values in $\mathcal{C}^\natural(\widehat{G})$ are isomorphic to solutions of $\widehat{\mathcal{M}}$ with values in $\mathcal{C}^\natural(G)$, for $\natural = \pm\infty$, $\pm\omega$. This shows that, in order to deal with concrete examples, one has to address three problems of an independent nature

- (T) the computation of $\widehat{G}$,
- (A) the computation of $\widehat{\mathcal{M}}$,
- (Q) the explicit description of the isomorphism between solutions.

Problem (T) is the easiest one, since it is of a topological nature. Problem (A) is of an analytical nature, and we will explain how the theory of microlocal operators helps in dealing with it. In particular, under certain assumptions on the microlocal geometry associated to the correspondence (which are satisfied by the Radon transform), the functor $\mathcal{M} \mapsto \widehat{\mathcal{M}}$ establishes an equivalence of categories between coherent $\mathcal{D}$-modules on X and systems with regular singularities along an involutive submanifold of the cotangent bundle to Ξ, determined by the correspondence. Finally, (Q) is a kind of quantization problem, which consists in exhibiting the integral kernel of the transform. This is obtained as boundary value of the meromorphic kernel associated to the $\mathcal{D}$-module transform, which may be viewed as a complex analogue of Lagrangian distributions.

In the third and last section of our survey, we exemplify the above approach by giving proofs of the results on range characterization for Radon transforms. These are taken from joint works with P. Schapira, with C. Marastoni, or by the present author alone. Although we restricted our attention to classical results, it should already be evident that the tools of sheaf and $\mathcal{D}$-module theory are quite powerful in this context. In particular, they allow one to treat higher cohomology groups, obtaining results which are hardly accessible by classical means. As an example, we get a geometrical interpretation of the Cavalieri condition, which characterizes the image of the real affine Radon transform.

Let us make some comments on related results. Independently, several people proposed approaches to integral geometry which are more or less close to ours. A. B. Goncharov gave a $\mathcal{D}$-module interpretation of Gelfand's κ-form, used to describe inversion formulae. Kashiwara-Schmid announced an adjunction formula similar to ours, taking into account topology and group actions. In an equivariant setting, the computation of the $\mathcal{D}$-module transform for correspondences between

generalized flag manifolds appear in recent works by T. Oshima and T. Tanisaki. The computations in Baston-Eastwood should also be useful in this context.

Concerning the exposition that follows, to keep it lighter we decided to postpone the bibliographical comments to the end of each section. As a consequence, a lack of reference does not imply that a result should be attributed to the present author.

1. Radon transform(s)

Here, we collect some more or less classical results on range characterization for various instances of the Radon p-plane transform. This should be considered as a motivation for the theory we present in the next section.

1.1. Real affine case. The spaces

$$(1.1) \qquad \begin{cases} \mathsf{A} & \text{a real } n\text{-dimensional affine space,} \\ \mathsf{G}_\mathsf{A} & \text{the family of affine } p\text{-planes in } \mathsf{A}, \end{cases}$$

are related by the correspondence $\mathsf{G}_\mathsf{A} \ni \xi \multimap \widehat{\xi} \subset \mathsf{A}$, associating to a plane the set of its points. Denote by $\mathcal{S}(\mathsf{A})$ the Schwartz space of rapidly decreasing C^∞-functions on A. The classical affine Radon transform consists in associating to a function φ in $\mathcal{S}(\mathsf{A})$ its integrals along the family of p-planes

$$(1.2) \qquad R_\mathsf{A} \colon \mathcal{S}(\mathsf{A}) \ni \varphi \mapsto \psi(\xi) = \int_{\widehat{\xi}} \varphi.$$

In order to explain what measure is used in the above integral, let us choose a system of coordinates $(t) = (t_1, \ldots, t_n)$ in $\mathsf{A} \simeq \mathbb{R}^n$. Any affine p-plane $\xi \in \mathsf{G}_\mathsf{A}$ can be described by a system of equations

$$\xi \colon \langle t, \tau_i \rangle + \sigma_i = 0, \qquad i = 1, \ldots, n - p,$$

where $\tau_i = (\tau_i^1, \ldots, \tau_i^n)$ are linearly independent vectors in $(\mathbb{R}^n)^*$, and $\sigma_i \in \mathbb{R}$. Consider

$$\psi(\sigma, \tau) = \int \varphi(t)\delta(\langle t, \tau_1 \rangle + \sigma_1) \cdots \delta(\langle t, \tau_{n-p} \rangle + \sigma_{n-p}) \, dt_1 \cdots dt_n,$$

where $\sigma \in \mathbb{R}^{n-p}$ is the row vector $\sigma = (\sigma_1, \ldots, \sigma_{n-p})$, $\tau = (\tau_1, \ldots, \tau_{n-p})$ is an $n \times (n-p)$ matrix, and δ denotes the Dirac delta function. Since ψ satisfies the homogeneity condition

$$\psi(\sigma\mu, \tau\mu) = |\det \mu|^{-1}\psi(\sigma, \tau) \quad \forall \mu \in \mathrm{GL}(n - p, \mathbb{R}),$$

it defines a section of a line bundle over G_A, that we denote by $\mathcal{C}^\infty_{\mathsf{G}_\mathsf{A}}\{-1\}$. The projection

$$(\xi \colon \langle t, \tau_i \rangle + \sigma_i = 0) \mapsto (\xi' \colon \langle t, \tau_i \rangle = 0)$$

makes G_A into an $(n - p)$-dimensional vector bundle $q \colon \mathsf{G}_\mathsf{A} \to \mathsf{G}'$ over the compact Grassmannian of vector p-planes in $\mathbb{R}^n$. One says that a global section of $\mathcal{C}^\infty_{\mathsf{G}_\mathsf{A}}\{-1\}$ is rapidly decreasing, if it is rapidly decreasing along the fibers of q.

THEOREM 1.1. *The real affine Radon transform*

$$R_\mathsf{A} \colon \mathcal{S}(\mathsf{A}) \to \Gamma(\mathsf{G}_\mathsf{A}; \mathcal{C}^\infty_{\mathsf{G}_\mathsf{A}}\{-1\})$$

$$\varphi(t) \mapsto \psi(\sigma, \tau)$$

is injective. Concerning its range

(i) *If $p < n - 1$, then ψ belongs to the image of R_{A} if and only if it is rapidly decreasing and satisfies the John ultrahyperbolic system*

$$(1.3) \qquad \left(\frac{\partial^2}{\partial \xi_i^j \partial \xi_{i'}^{j'}} - \frac{\partial^2}{\partial \xi_i^{j'} \partial \xi_{i'}^{j}} \right) \psi(\xi) = 0, \qquad \begin{array}{l} j, j' = 0, \ldots, n, \\ i, i' = 1, \ldots, n - p, \end{array}$$

where $(\xi) = (\xi_1, \ldots, \xi_{n-p})$, $\xi_i = (\xi_i^0, \ldots \xi_i^n) = (\sigma_i, \tau_i^1, \ldots, \tau_i^n)$.

(ii) *If $p = n - 1$, then ψ belongs to the image of R_{A} if and only if it is rapidly decreasing and satisfies the Cavalieri condition*

$$\int_{-\infty}^{+\infty} \psi(\sigma, \tau) \sigma^m \, d\sigma \quad \text{is, for any } m \in \mathbb{N}, \text{ a polynomial in } \tau \text{ of degree} \leq m.$$

(Since $n - p = 1$, we have here $\sigma = \sigma_1$ and $\tau = \tau_1$.)

Heuristically, the system of PDE in (i) compensates for the difference of dimensions between $\dim \mathsf{A} = n$ and $\dim \mathsf{G}_{\mathsf{A}} = (p+1)(n-p)$. Also for $p = n - 1$ some conditions should have been expected. In fact, $\mathcal{S}(\mathsf{A})$ is the space of functions which are rapidly decreasing in the $n - 1$ directions of A going to infinity, while rapid decrease in G_{A} concerns only the 1-dimensional fiber of $q \colon \mathsf{G}_{\mathsf{A}} \to \mathsf{G}'$. The Cavalieri condition—also known as moment condition—was classically obtained using Fourier inversion formula. Instead, using our formalism, we will obtain this condition in a purely geometric way.

1.2. Complex projective case. Let us consider

$$(1.4) \qquad \begin{cases} \mathbb{V} & \text{a complex vector space of dimension } n + 1, \\ \mathbb{P} & \text{the projective space of complex vector lines in } \mathbb{V}, \\ \mathbb{G} & \text{the Grassmannian of projective } p\text{-planes in } \mathbb{P}. \end{cases}$$

As for the real case, there is a correspondence $\mathbb{G} \ni \zeta \multimap \widehat{\zeta} \subset \mathbb{P}$, associating to a plane the set of its points. Reciprocally, denote by $\widehat{z} \subset \mathbb{G}$ the set of p-planes containing $z \in \mathbb{P}$. We also set $\widehat{U} = \bigcup_{\zeta \in U} \widehat{\zeta}$ for $U \subset \mathbb{G}$.

If $[z] = [z_0, \ldots, z_n]$ are homogeneous coordinates in $\mathbb{P}$, a projective p-plane is described by the system of equations

$$(1.5) \qquad\qquad \zeta \colon \langle z, \zeta_i \rangle = 0, \qquad i = 1, \ldots, n - p,$$

where ζ_i are linearly independent vectors in $\mathbb{V}^*$.

For $m \in \mathbb{Z}$, let us denote by $\mathcal{O}_{\mathbb{P}}(m)$ the holomorphic line bundle whose sections φ, written in homogeneous coordinates, satisfy the homogeneity condition

$$\varphi(\lambda z) = \lambda^m \varphi(z) \quad \forall \lambda \in \mathbb{C}^\times = \mathrm{GL}(1, \mathbb{C}).$$

The Leray form

$$\omega(z) = \sum_{j=0}^{n} (-1)^j z_j dz_0 \wedge \cdots \wedge dz_{j-1} \wedge dz_{j+1} \wedge \cdots \wedge dz_n$$

is $(n+1)$-homogeneous, i.e. is a global section of $\Omega_{\mathbb{P}}(n+1) = \Omega_{\mathbb{P}} \otimes_{\mathcal{O}_{\mathbb{P}}} \mathcal{O}_{\mathbb{P}}(n+1)$, where $\Omega_{\mathbb{P}}$ denotes the sheaf of holomorphic forms of maximal degree.

Let $U \subset \mathbb{G}$ be an open subset such that $\widehat{z} \cap U$ is connected for any $z \in \widehat{U}$. Consider the integral transform

$$(1.6) \qquad R_{\mathbb{P}} \colon H^p(\widehat{U}; \mathcal{O}_{\mathbb{P}}(-p-1)) \ni \varphi \mapsto \psi(\zeta) = \left(\frac{1}{2\pi i} \right)^{n-p} \int \varphi(z) \frac{\omega(z)}{\langle z, \zeta_1 \rangle \cdots \langle z, \zeta_{n-p} \rangle},$$

where the integral sign stands for a Leray-Grothendieck residue. Note that the integrand is a well defined function in the $\mathbb{P}$ variable, since it is homogeneous of degree zero in z. Moreover, $\psi(\zeta) \in \Gamma(U; \mathcal{O}_{\mathbb{G}}(-1))$, where $\mathcal{O}_{\mathbb{G}}(-1)$ is the line bundle whose sections satisfy the homogeneity condition

$$\psi(\zeta\mu) = (\det \mu)^{-1}\psi(\zeta) \quad \forall \mu \in \mathrm{GL}(n-p, \mathbb{C}).$$

Recall that, by the Cauchy formula, the Dirac delta function $\delta(u)$ in $\mathbb{R}$ is the boundary value of $(2\pi i\, w)^{-1}$, for $w = u + iv \in \mathbb{C}$. Thus, the integral kernel of $R_{\mathbb{P}}$ is the complex analog of the delta function along the incidence relation $z \in \widehat{\zeta}$. In this sense $R_{\mathbb{P}}$ is indeed a complex Radon transform $\varphi \mapsto \int_{\widehat{\zeta}} \varphi$.

Following Penrose, if $p = 1$ and $n = 3$ then $\mathbb{G} = \mathbb{M}$ is a conformal compactification of the linear complexified Minkowski space. Maxwell's wave equation thus extends to an equivariant differential operator on $\mathbb{M}$. For arbitrary $p < n - 1$, we denote by $\square$ its higher dimensional analog in $\mathbb{G}$

(1.7) $$\square \colon \mathcal{O}_{\mathbb{G}}(-1) \to \mathcal{H},$$

where $\mathcal{H}$ is a homogeneous vector bundle that we do not need to describe here. Let us just point out that $\square$ is a complex version of the John ultrahyperbolic system. In fact, $\square\psi(\zeta)$ is expressed by the left hand side term of (1.3), replacing (ξ_i^j) with the dual Stiefel coordinates of $\mathbb{G}$, $(\zeta_i^j) \in (\mathbb{V}^*)^{n-p}$.

One says that $U \subset \mathbb{G}$ is elementary if for any $z \in \widehat{U}$ the slice $\widehat{z} \cap U$ has trivial reduced cohomology up to degree p. This means that $\widehat{z} \cap U$ should be connected, and its Betti numbers b^j should vanish for $1 \le j \le p$.

THEOREM 1.2. *Let U be an elementary open subset of $\mathbb{G}$, and consider the complex Radon transform*

$$R_{\mathbb{P}} \colon H^p(\widehat{U}; \mathcal{O}_{\mathbb{P}}(-p-1)) \to \Gamma(U; \mathcal{O}_{\mathbb{G}}(-1)).$$

(i) *If $p < n - 1$, then $R_{\mathbb{P}}$ is an isomorphism onto the space of sections ψ satisfying $\square\psi = 0$.*
(ii) *If $p = n - 1$, then $\mathbb{G} = \mathbb{P}^*$ is a dual projective space, and $R_{\mathbb{P}}$ is an isomorphism.*

1.3. Real conformal case. For simplicity sake, let us restrict our attention to a generalization of the Penrose case, where $n + 1 = 2(p + 1)$. Consider the following geometrical situation

(1.8) $$\begin{cases} \Phi & \text{a Hermitian form on } \mathbb{V} \simeq \mathbb{C}^{2(p+1)} \text{ of signature } (p+1, p+1), \\ \mathsf{Q} \subset \mathbb{P} & \text{the set of null vectors } z, \text{ with } \Phi|_z = 0, \\ \mathsf{M} \subset \mathbb{G} & \text{the set of null planes } \zeta, \text{ with } \Phi|_\zeta = 0. \end{cases}$$

One checks that Q is a real hypersurface of $\mathbb{P}$ whose Levi form has rank $2p$ and signature (p, p), and M is a totally real analytic submanifold of $\mathbb{G}$. Let us denote by $\mathcal{C}_{\mathsf{M}}^{\omega}$ and $\mathcal{C}_{\mathsf{M}}^{-\omega}$ the sheaves of real analytic functions and Sato hyperfunctions on M, respectively. Since the higher dimensional Maxwell wave operator (1.7) is hyperbolic for M, it is natural to consider its hyperfunction solutions. For $\natural = \pm\omega$, denote by $\ker(\square; \mathcal{C}_{\mathsf{M}}^{\natural})$ the subsheaf of $\mathcal{C}_{\mathsf{M}}^{\natural} \otimes_{\mathcal{O}_{\mathbb{G}}} \mathcal{O}_{\mathbb{G}}(-1)$ whose sections ψ satisfy $\square\psi = 0$.

One has the following boundary value version of Theorem 1.2.

THEOREM 1.3. *There is a commutative diagram*

$$\begin{array}{ccc} H^p(\mathsf{Q};\mathcal{O}_{\mathbb{P}(-p-1)}) & \xrightarrow{\sim} & \Gamma(\mathsf{M};\ker(\square;\mathcal{C}_\mathsf{M}^\omega)) \\ \downarrow & & \downarrow \\ H_\mathsf{Q}^{p+1}(\mathbb{P};\mathcal{O}_{\mathbb{P}(-p-1)}) & \xrightarrow{\sim} & \Gamma(\mathsf{M};\ker(\square;\mathcal{C}_\mathsf{M}^{-\omega})) \end{array}$$

where the horizontal arrows are induced by $R_\mathbb{P}$, the vertical arrows are the natural maps, and $H_\mathsf{Q}^j(\cdot)$ denotes the j-th cohomology group with support on Q.

Notes. We give here some references to papers where the Radon transform is dealt with using a classical approach, i.e. without using the tools of sheaf and $\mathcal{D}$-module theory.

§1.1 The study of the hyperplane Radon transform in affine 3-space goes back to Radon himself [Rad17]. Line integrals were considered by John [Joh38], where the ultrahyperbolic equation first appeared. Theorem 1.1 may be found, for example, in [GGG82] or [Hel84]. See also [Gri85], [Gnz91], and [Kak97] for related results.

§1.2 Part (ii) of Theorem 1.2 is due to Martineau [Mar67], while part (i) for $p = 1$, $n = 3$ is known as the Penrose transform, and was obtained in [EPW81]. Correspondences between generalized flag manifolds are discussed in [BE89]. Theorem 1.2 is also considered by [GH78], [HP87], in the case where $\widehat{U}$ is p-linearly concave. The case where $n + 1 = 2(p + 1)$ and $U = \{\zeta \colon \Phi|_\zeta \gg 0\}$ is discussed in [Sek96], and generalized to other correspondences between Grassmannians, using representation theoretical arguments.

§1.3 Theorem 1.3 for $p = 1$ is due to [Wel81]. Note also that the problem of extending to M sections of $\ker(\square;\mathcal{C}_\mathsf{M}^{-\omega})$ defined on the affine Minkowski space is discussed in [BEW82].

2. Sheaves and $\mathcal{D}$-modules for integral geometry

There is a common pattern underlying the examples we discussed in the previous section, which the language of sheaves and $\mathcal{D}$-modules will make apparent. As a motivation for our choice of framework, let us state an informal paradigm for integral transforms.

Let X be a manifold, denote by $\mathcal{F}$ a space of "functions" on X, let $\mathcal{M}$ represent some system of PDE (possibly void, since the absence of differential equations corresponds to the system $0u = 0$), and denote by $\mathrm{Sol}(\mathcal{M},\mathcal{F})$ the space of $\mathcal{F}$-valued solutions to $\mathcal{M}$. Similarly, consider the space $\mathrm{Sol}(\mathcal{N},\mathcal{G})$ on another manifold Ξ. An integral transform from X to Ξ is a map

(2.1)
$$\begin{array}{rcl} \mathrm{Sol}(\mathcal{M},\mathcal{F}) & \to & \mathrm{Sol}(\mathcal{N},\mathcal{G}) \\[1mm] \varphi(x) & \mapsto & \psi(\xi) = \displaystyle\int \varphi(x) \cdot k(x,\xi), \end{array}$$

consisting of the composition of three operations

- (i) pull back φ to $X \times \Xi$,
- (ii) take the product with an integral kernel k,
- (iii) push it forward to Ξ.

To make some sense out of this, we have to decide what we mean by space of functions, how we represent systems of PDE, and how we perform the three steps above in this setting.

2.1. Generalized functions. Let X be a complex manifold, and denote by $\mathcal{O}_X$ its structural sheaf of holomorphic functions. Starting from $\mathcal{O}_X$, some spaces of functions are naturally associated to the datum of a locally closed subanalytic subset.

EXAMPLES. • To an open subset $U \subset X$ is associated the space $\Gamma(U; \mathcal{O}_X)$ of holomorphic functions on U. More generally, one may consider the cohomology groups $H^j(U; \mathcal{O}_X)$.
• To a real analytic submanifold M, of which X is a complexification, are associated the sheaves

$$(2.2) \qquad \mathcal{C}_M^\omega \subset \mathcal{C}_M^\infty \subset \mathcal{C}_M^{-\infty} \subset \mathcal{C}_M^{-\omega}$$

of real analytic functions, C^∞-functions, Schwartz distributions and Sato hyperfunctions, respectively. Recall that, locally, hyperfunctions are represented by finite sums $\sum_j b(F_j)$, where $b(F_j)$ denotes the boundary value of a holomorphic function F_j defined on an open wedge of X with edge M. For example, hyperfunctions on the real line are equivalence classes in $\Gamma(\mathbb{C} \setminus \mathbb{R}; \mathcal{O}_X)/\Gamma(\mathbb{C}; \mathcal{O}_X)$. More precisely, $\mathcal{C}_M^{-\omega} = H_M^{\dim X}(\mathcal{O}_X) \otimes \mathrm{or}_{M/X}$ is obtained from $\mathcal{O}_X$ by considering cohomology with support in M, twisted by the relative orientation sheaf $\mathrm{or}_{M/X}$ (since $\mathrm{or}_{M/X}$ is locally trivial, the twisting operation is locally void).

The set of spaces as above, obtained from $\mathcal{O}_X$ via the datum of a subanalytic subset, is not stable by direct images. For example

• Let $X = \mathbb{C} \setminus \{0\}$ with holomorphic coordinate z, and define $p \colon X \to X$ by $p(z) = z^2$. Then $\sqrt{z} \in p_* \mathcal{O}_X$ is described by a local system rather than by a subset. In fact, one has $p_* \mathcal{O}_X \simeq \hom(p_* \mathbb{C}_X, \mathcal{O}_X)$, and $\sqrt{z} \in \hom(L, \mathcal{O}_X)$, where $\mathbb{C}_X$ denotes the constant sheaf on X, and L is defined by $p_* \mathbb{C}_X \simeq \mathbb{C}_X \oplus L$.

Instead of subanalytic subsets, we thus consider the bounded derived category $\mathbf{D}_{\mathbb{R}-\mathrm{c}}^{\mathrm{b}}(\mathbb{C}_X)$ of $\mathbb{R}$-constructible sheaves. Roughly speaking, this is the smallest full subcategory of the bounded derived category of sheaves such that: (i) it contains the skyscraper sheaves $\mathbb{C}_U$ along open subanalytic subsets $U \subset X$, (ii) the assignment $X \mapsto \mathbf{D}_{\mathbb{R}-\mathrm{c}}^{\mathrm{b}}(\mathbb{C}_X)$ is stable by exterior tensor products, inverse images, and proper direct images.
With these notations, some of the above examples are expressed as

$$H^j(U; \mathcal{O}_X) = H^j \mathrm{RHom}(\mathbb{C}_U, \mathcal{O}_X), \qquad \sqrt{z} \in \hom(L, \mathcal{O}_X),$$
$$\mathcal{C}_M^\omega = \mathbb{C}_M \otimes \mathcal{O}_X, \qquad \mathcal{C}_M^{-\omega} = \mathrm{Rhom}(\mathbb{C}'_M, \mathcal{O}_X),$$

where we denote by $F' = \mathrm{Rhom}(F, \mathbb{C}_X)$ the dual of $F \in \mathbf{D}_{\mathbb{R}-\mathrm{c}}^{\mathrm{b}}(\mathbb{C}_X)$.
It is also possible to obtain in a similar way the sheaves of C^∞-functions and Schwartz distributions, replacing $\otimes$ and Rhom by the functors $\overset{\mathrm{w}}{\otimes}$ and $t$hom of formal and tempered cohomology. Let us briefly recall their construction. First, using a result of Lojasiewicz, one proves that there exist exact functors w and t in $\mathbf{D}_{\mathbb{R}-\mathrm{c}}^{\mathrm{b}}(\mathbb{C}_X)$, characterized by the following requirements. If S is a closed subanalytic subset of X, then $w(\mathbb{C}_{X\setminus S})$ is the ideal of $\mathcal{C}_X^\infty$ of functions vanishing to infinite order on S, and $t(\mathbb{C}_S)$ is the subsheaf of $\mathcal{C}_X^{-\infty}$ whose sections have support contained in S. Then, for $F \in \mathbf{D}_{\mathbb{R}-\mathrm{c}}^{\mathrm{b}}(\mathbb{C}_X)$ one defines $F \overset{\mathrm{w}}{\otimes} \mathcal{O}_X$ and $t$hom$(F, \mathcal{O}_X)$ as the Dolbeault complexes with coefficients in $w(F)$ and $t(F)$, respectively.

To $F \in \mathbf{D}^{\mathrm{b}}_{\mathbb{R}-\mathrm{c}}(\mathbb{C}_X)$ we thus associate the spaces of "generalized functions"

$$\mathcal{C}^{\natural}(F) = \begin{cases} F \otimes \mathcal{O}_X & \natural = \omega, \\ F \overset{\mathrm{w}}{\otimes} \mathcal{O}_X & \natural = \infty, \\ \mathrm{thom}(F', \mathcal{O}_X) & \natural = -\infty, \\ \mathrm{Rhom}(F', \mathcal{O}_X) & \natural = -\omega. \end{cases}$$

In particular, (2.2) is obtained from the natural sequence

$$\mathcal{C}^{\omega}(F) \to \mathcal{C}^{\infty}(F) \to \mathcal{C}^{-\infty}(F) \to \mathcal{C}^{-\omega}(F)$$

by taking $F = \mathbb{C}_M$.

2.2. Systems of PDE. Denote by $\mathcal{D}_X$ the sheaf of linear differential operators with holomorphic coefficients in X. By definition, a (left) $\mathcal{D}_X$-module $\mathcal{M}$ is coherent if it is locally represented by a system of PDE, i.e. if locally there exists an exact sequence

$$(\mathcal{D}_X)^{N_1} \xrightarrow[\cdot P]{} (\mathcal{D}_X)^{N_0} \to \mathcal{M} \to 0,$$

where $P = (P_{ij})$ is an $N_1 \times N_0$ matrix with elements in $\mathcal{D}_X$, acting by multiplication to the right $Q \mapsto QP$. We denote by $\mathbf{D}^{\mathrm{b}}_{\mathrm{coh}}(\mathcal{D}_X)$ the bounded derived category of $\mathcal{D}_X$-modules with coherent cohomology groups. To $\mathcal{M} \in \mathbf{D}^{\mathrm{b}}_{\mathrm{coh}}(\mathcal{D}_X)$ is associated its characteristic variety $\mathrm{char}(\mathcal{M})$, a closed complex subvariety of the cotangent bundle T^*X, which is involutive for the natural symplectic structure.

If $\mathcal{F}$ is another $\mathcal{D}_X$-module (not necessarily coherent), the solutions of $\mathcal{M}$ with values in $\mathcal{F}$ are obtained by

$$\mathrm{sol}(\mathcal{M}, \mathcal{F}) = \mathrm{Rhom}_{\mathcal{D}_X}(\mathcal{M}, \mathcal{F}),$$

and we will also consider the space of global sections

$$\mathrm{Sol}(\mathcal{M}, \mathcal{F}) = \mathrm{R}\Gamma(X; \mathrm{sol}(\mathcal{M}, \mathcal{F})).$$

EXAMPLES. • To a holomorphic vector bundle $\mathcal{H}$ one associates the $\mathcal{D}_X$-module $\mathcal{D}\mathcal{H}^* = \mathcal{D}_X \otimes_{\mathcal{O}_X} \mathcal{H}^*$, where $\mathcal{H}^* = \mathrm{hom}_{\mathcal{O}_X}(\mathcal{H}, \mathcal{O}_X)$. Since $\mathcal{D}\mathcal{H}^*$ is locally represented by the trivial system $0u = 0$, its characteristic variety is the whole space T^*X. We can recover the $\mathbb{C}$-module underlying $\mathcal{H}$ by taking holomorphic solutions to $\mathcal{D}\mathcal{H}^*$, namely $\mathcal{H} \simeq \mathrm{sol}(\mathcal{D}\mathcal{H}^*, \mathcal{O}_X)$.

• Next, consider the cyclic $\mathcal{D}_X$-module $\mathcal{D}^P_X = \mathcal{D}_X/\mathcal{D}_X P$ associated to a single differential operator $P \in \mathcal{D}_X$. Its characteristic variety is the hypersurface defined by the zero locus of the principal symbol of P. Concerning its solutions, $H^0 \mathrm{sol}(\mathcal{D}^P_X, \mathcal{F}) = \{\varphi \in \mathcal{F} : P\varphi = 0\}$ describes the kernel of the operator P acting on $\mathcal{F}$, and $H^1 \mathrm{sol}(\mathcal{D}^P_X, \mathcal{F}) = \mathcal{F}/P\mathcal{F}$ describes the obstruction to solve the inhomogeneous equation $P\varphi = \psi$.

• Combining the two examples above, let $\nabla \colon \mathcal{G} \to \mathcal{H}$ be a differential operator acting between two holomorphic vector bundles. Using the identification $\mathrm{Homdiff}(\mathcal{G}, \mathcal{H}) \simeq \mathrm{Hom}_{\mathcal{D}_X}(\mathcal{D}\mathcal{H}^*, \mathcal{D}\mathcal{G}^*)$, we associate to ∇ the $\mathcal{D}_X$-module $\mathcal{D}^{\nabla}_X$ represented by

$$\mathcal{D}\mathcal{H}^* \to \mathcal{D}\mathcal{G}^* \to \mathcal{D}^{\nabla}_X \to 0.$$

One has $H^0 \mathrm{sol}(\mathcal{D}^{\nabla}_X, \mathcal{O}_X) = \ker \nabla$.

• Recall that a $\mathcal{D}_X$-module is called holonomic if its characteristic variety has the smallest possible dimension, i.e. is a Lagrangian subvariety of T^*X. In other

words, holonomic modules are locally associated to maximally overdetermined systems of PDE. If $\mathcal{M}$ is holonomic, then $\mathrm{sol}(\mathcal{M}, \mathcal{O}_X)$ is $\mathbb{C}$-constructible. On the other hand, if F is $\mathbb{C}$-constructible, then $\mathit{thom}(F, \mathcal{O}_X)$ has regular holonomic cohomology groups. (Regularity is a generalization to higher dimensions of the notion of Fuchsian ordinary differential operators.) More precisely, the Riemann-Hilbert correspondence asserts that the functors $\mathrm{sol}(\cdot, \mathcal{O}_X)$ and $\mathit{thom}(\cdot, \mathcal{O}_X)$ are quasi-inverse to each other.

Recall that to a d-codimensional submanifold $S \subset X$ one associates the regular holonomic $\mathcal{D}$-module of holomorphic hyperfunctions $\mathcal{B}_{S|X} = H^d \mathit{thom}(\mathbb{C}_S, \mathcal{O}_X)$. Using more classical notations, $\mathcal{B}_{S|X} = H^d_{[S]}(\mathcal{O}_X)$ is a cohomology group with tempered support. The characteristic variety of $\mathcal{B}_{S|X}$ is the conormal bundle $T^*_S X$. If S is a hypersurface, then elements of $\mathcal{B}_{S|X}$ are equivalence classes of meromorphic functions with poles along S, modulo holomorphic functions. Even more specifically, $\mathcal{B}_{\{0\}|\mathbb{C}}$ is generated by the equivalence class of $(2\pi i z)^{-1}$, where $z \in \mathbb{C}$ is the holomorphic coordinate.

2.3. Adjunction formula. In view of the above discussion, we consider the following setting for (2.1)

★ X is a complex manifold	★ Ξ is a complex manifold
★ $\mathcal{M} \in \mathbf{D}^{\mathrm{b}}_{\mathrm{coh}}(\mathcal{D}_X)$	★ $\mathcal{N} \in \mathbf{D}^{\mathrm{b}}_{\mathrm{coh}}(\mathcal{D}_\Xi)$
★ $F \in \mathbf{D}^{\mathrm{b}}_{\mathbb{R}-c}(\mathbb{C}_X)$	★ $G \in \mathbf{D}^{\mathrm{b}}_{\mathbb{R}-c}(\mathbb{C}_\Xi)$
★ $\mathcal{F} = \mathcal{C}^\natural(F)$ for $\natural = \pm\omega, \pm\infty$	★ $\mathcal{G} = \mathcal{C}^\natural(G)$ for $\natural = \pm\omega, \pm\infty$

Consider the projections

$$X \xleftarrow{q_1} X \times \Xi \xrightarrow{q_2} \Xi.$$

The three operations of pull-back, product, and push-forward, which appear in an expression like $\int \varphi(x) \cdot k(x, \xi) = \int_{q_2} (q_1^* \varphi \cdot k)$, make sense in the categories of sheaves and $\mathcal{D}$-modules. We denote them by

$$q_1^{-1}, \; \otimes, \; Rq_{2!}, \qquad \text{and} \qquad Dq_1^*, \; \overset{D}{\otimes}, \; Dq_{2!},$$

respectively. For $\mathcal{D}$-modules, the analog of $\int_{q_2} (q_1^* \varphi \cdot k)$ is

$$\mathcal{M} \overset{D}{\circ} \mathcal{K} = Dq_{2!}(Dq_1^* \mathcal{M} \overset{D}{\otimes} \mathcal{K}),$$

where $\mathcal{M}$ and $\mathcal{K}$ are (complexes of) $\mathcal{D}$-modules on X and on $X \times \Xi$, respectively. Similarly, to the (complexes of) sheaves G on Ξ and K on $X \times \Xi$, we associate the sheaf on X

$$K \circ G = Rq_{1!}(q_2^{-1} G \otimes K).$$

Assume that K and $\mathcal{K}$ are interchanged by the Riemann-Hilbert correspondence, i.e. assume one of the equivalent conditions

★ K is $\mathbb{C}$-constructible and $\mathcal{K} = \mathit{thom}(K, \mathcal{O}_{X \times \Xi})$,
★ $\mathcal{K}$ is regular holonomic and $K = \mathrm{sol}(\mathcal{K}, \mathcal{O}_{X \times \Xi})$.

In order to ensure that $\mathbb{R}$-constructibility is preserved by the functor $K \circ \cdot$, and coherence is preserved by the functor $\cdot \overset{D}{\circ} \mathcal{K}$, let us also assume that

★ q_1 and q_2 are proper on $\mathrm{supp}(\mathcal{K})$,
★ $\mathcal{K}$ is transversal to $Dq_1^* \mathcal{M}$, for any $\mathcal{M}$,

where the latter condition means that $\operatorname{char}(\mathcal{K}) \cap (T^*X \times T^*_\Xi \Xi)$ is contained in the zero-section of $T^*(X \times \Xi)$.

Denote as usual by $(\cdot)[1]$ the shift functor in derived categories.

THEOREM 2.1. *With the above notations and hypotheses ★, we have the adjunction formula*

$$(2.3) \qquad \operatorname{Sol}(\mathcal{M}, \mathcal{C}^\natural(K \circ G))[\dim X] \simeq \operatorname{Sol}(\mathcal{M} \overset{D}{\circ} \mathcal{K}, \mathcal{C}^\natural(G)).$$

In particular, to the pair of a $\mathcal{D}$-linear morphism $\alpha \colon \mathcal{N} \to \mathcal{M} \overset{D}{\circ} \mathcal{K}$ and a $\mathbb{C}$-linear morphism $\beta \colon F \to K \circ G$ is associated a morphism

$$\operatorname{Sol}(\mathcal{M}, \mathcal{C}^\natural(F))[\dim X] \underset{\beta \cdot \alpha}{\longrightarrow} \operatorname{Sol}(\mathcal{N}, \mathcal{C}^\natural(G)).$$

This is the archetypical theorem we referred to in the Abstract. In order to deal with concrete examples, one has to address three problems, which are of an independent nature:

(T) compute the sheaf transform $K \circ G$,
(A) compute the $\mathcal{D}$-module transform $\mathcal{M} \overset{D}{\circ} \mathcal{K}$,
(Q) identify the morphism $\beta \cdot \alpha$.

Problem (T) is purely topological, problem (A) is of an analytic nature, and (Q) is a quantization problem. Usually the most difficult of the three is problem (A). In the classical literature it corresponds to finding the system of PDE that characterizes the image of the transform. This problem is split into two parts, of which the second is much more difficult

(i) find differential equations satisfied by functions in the image of the transform,
(ii) prove that these equations are sufficient to characterize the image.

In our language, this translates into

(i) find a $\mathcal{D}$-linear morphism $\mathcal{N} \to \mathcal{M} \overset{D}{\circ} \mathcal{K}$,
(ii) prove it is an isomorphism.

An answer to the first issue is given by the following lemma. Although its proof is quite straightforward, it plays a key role.

LEMMA 2.2. *With the above notations and hypotheses ★, one has an isomorphism*

$$\alpha \colon H^0 \operatorname{Sol}(\mathcal{M}^\vee \overset{D}{\boxtimes} \mathcal{N}, \mathcal{K}) \overset{\sim}{\longrightarrow} \operatorname{Hom}_{\mathcal{D}_\Xi}(\mathcal{N}, \mathcal{M} \overset{D}{\circ} \mathcal{K}),$$

where $\mathcal{M}^\vee = \hom_{\mathcal{D}_X}(\mathcal{M} \otimes_{\mathcal{O}_X} \Omega_X, \mathcal{D}_X)$ is the dual of $\mathcal{M}$ as left $\mathcal{D}_X$-modules, Ω_X denotes the sheaf of holomorphic forms of maximal degree, and $\operatorname{Hom}_{\mathcal{D}_\Xi}(\cdot, \cdot)$ is the set of globally defined $\mathcal{D}_\Xi$-linear morphisms.

This shows that any morphism $\mathcal{N} \to \mathcal{M} \overset{D}{\circ} \mathcal{K}$ is of the form $\alpha(\kappa)$, where κ is a globally defined $\mathcal{K}$-valued solution of the system $\mathcal{M}^\vee$ in the X variables, and of the system $\mathcal{N}$ in the Ξ variables. In other words, to any morphism $\mathcal{N} \to \mathcal{M} \overset{D}{\circ} \mathcal{K}$ is attached an integral kernel κ. We will show in Proposition 2.4 that, under some microlocal assumptions, one may as well read off from κ the condition for $\alpha(\kappa)$ to be an isomorphism.

Concerning problem (Q), note that the morphism $\beta \cdot \alpha$ is obtained as the composite

$$
\begin{array}{rcl}
\mathrm{Sol}(\mathcal{M}, \mathcal{C}^{\natural}(F))[\dim X] & \rightarrow & \mathrm{Sol}(\mathcal{M}, \mathcal{C}^{\natural}(K \circ G))[\dim X] \\
& \simeq & \mathrm{Sol}(\mathcal{M} \overset{D}{\circ} \mathcal{K}, \mathcal{C}^{\natural}(G)) \\
& \rightarrow & \mathrm{Sol}(\mathcal{N}, \mathcal{C}^{\natural}(G)),
\end{array}
$$

where the first and last morphisms are naturally induced by β and α, respectively. One then has to solve the quantization problem of describing the distribution kernel k of the integral transform $\beta \cdot \alpha$. (Such kernels may be considered as the analog of Lagrangian distributions for Fourier Integral Operators.) In view of Lemma 2.2, one should obtain k as a boundary value of κ, β describing the boundary value operation. An example of this phenomenon appears in Section 3.5, where we deal with the real projective Radon transform.

2.4. Integral geometry. In integral geometry one is given the graph $S \subset X \times \Xi$ of a correspondence from Ξ to X. Considering the natural morphisms

$$
\text{(2.4)} \qquad\qquad\qquad X \underset{f}{\leftarrow} S \underset{g}{\rightarrow} \Xi
$$

induced by q_1 and q_2, the correspondence $\Xi \ni \xi \multimap \widehat{\xi} \subset X$ is given by $\widehat{\xi} = f(g^{-1}(\xi))$. For simplicity, let us assume that

★ S is a complex submanifold of $X \times \Xi$,
★ f and g are smooth and proper.

One is interested in those transforms whose integral kernel k is some characteristic class of S, so that $\int_{q_2}(q_1^*\varphi \cdot k) = \int_{\widehat{\xi}} \varphi$ becomes an integration along the family of subsets $\widehat{\xi}$. More precisely, Theorem 2.1 fits in the framework of integral geometry if $\mathcal{K}$ is a regular holonomic module whose characteristic variety coincides—at least outside of the zero section—with the conormal bundle $T_S^*(X \times \Xi)$.

In this context, a natural choice is $\mathcal{K} = \mathcal{B}_S$, where we use the shorthand notation $\mathcal{B}_S = \mathcal{B}_{S|X \times \Xi}$ for the sheaf of holomorphic hyperfunctions. The Riemann-Hilbert correspondence associates to $\mathcal{K} = \mathcal{B}_S$ the perverse complex $K = \mathbb{C}_S[-d]$, where $d = \mathrm{codim}_{X \times \Xi} S$. One easily checks that

$$
K \circ G \simeq Rf_! g^{-1} G[-d], \qquad \mathcal{M} \overset{D}{\circ} \mathcal{K} \simeq Dg_! Df^* \mathcal{M}.
$$

Let us show in which sense problem (T) of the previous section is purely topological. For example, let $G = \mathbb{C}_Z$ for $Z \subset \Xi$ a locally closed subanalytic subset. Then, the fiber of $K \circ G$ at $x \in X$ is computed by

$$
\begin{array}{rcl}
\text{(2.5)} \qquad (Rf_! g^{-1}\mathbb{C}_Z)_x & \simeq & (Rf_! \mathbb{C}_{g^{-1}(Z)})_x \\
& \simeq & R\Gamma_c(f^{-1}(x); \mathbb{C}_{f^{-1}(x) \cap g^{-1}(Z)}) \\
& \simeq & R\Gamma_c(\widehat{x}; \mathbb{C}_{\widehat{x} \cap Z}),
\end{array}
$$

where in the last isomorphism we used the identification $g \colon f^{-1}(x) \cap g^{-1}(Z) \xrightarrow{\sim} \widehat{x} \cap Z$. This shows that $K \circ G$ is determined by the (co)homology groups of the family of slices $x \multimap \widehat{x} \cap Z$.

2.5. Microlocal geometry. Let us consider the microlocal correspondence associated to (2.4)

$$(2.6) \qquad T^*X \xleftarrow{\ p_1\ } T^*_S(X \times \Xi) \xrightarrow{\ p_2^a\ } T^*\Xi,$$

where p_1 and p_2 are induced by the natural projections from $T^*(X \times \Xi)$, and $(\cdot)^a$ denotes the antipodal map of $T^*\Xi$. By restriction, this gives the correspondence

$$(2.7) \qquad \dot{T}^*X \xleftarrow{\ \dot{p}_1\ } \dot{T}^*_S(X \times \Xi) \xrightarrow{\ \dot{p}_2^a\ } \dot{T}^*\Xi,$$

where the dot means that we have removed the zero-sections.

Using the theory of microdifferential operators, we may get some *a priori* information on $\mathcal{M} \overset{D}{\circ} \mathcal{B}_S$.

PROPOSITION 2.3. *With the above notations and hypotheses* ★, *assume that the map $\dot{p}_2^a$ is finite. Then*

$$\mathrm{char}(\mathcal{M} \overset{D}{\circ} \mathcal{B}_S) \subset p_2^a(p_1^{-1}(\mathrm{char}(\mathcal{M}))).$$

Moreover, if $\mathcal{M}$ is concentrated in degree zero, then $H^j(\mathcal{M} \overset{D}{\circ} \mathcal{B}_S)$ is a flat connection for $j \neq 0$. Finally, if $\mathcal{H}$ is a holomorphic vector bundle, then $H^j(\mathcal{D}\mathcal{H}^* \overset{D}{\circ} \mathcal{B}_S) = 0$ for $j < 0$.

Assume now

★ $\dot{p}_1$ is smooth surjective,

★ $\dot{p}_2^a$ is a closed embedding onto a regular involutive submanifold $\dot{W} \subset \dot{T}^*\Xi$.

The above conditions imply that the fibers of p_1 are identified with the bicharacteristic leaves of $\dot{W}$. Thus, it is not difficult to prove that, locally on $\dot{T}^*_S(X \times \Xi)$, the correspondence (2.7) is isomorphic to a contact transformation with parameters.

Let us recall three notions from the theory of microdifferential operators. First, one says that a $\mathcal{D}_\Xi$-module $\mathcal{N}$ has simple characteristics along $\dot{W}$ if in a neighborhood of any point in $\dot{W}$, its associated microdifferential system admits a simple generator whose symbol ideal is reduced and coincides with the annihilating ideal of $\dot{W}$. In particular, if $\mathcal{H}$ is a line bundle on X, then $\mathcal{D}\mathcal{H}^*$ is simple along $\dot{T}^*X$. Second, there is a natural notion of non-degenerate section of $\mathcal{B}_S$, which is local on $\dot{T}^*_S(X \times \Xi)$. For example, if S is a smooth hypersurface with local equation $(\Phi = 0)$, then a section of $\mathcal{B}_S$ is non-degenerate if it is obtained by applying an invertible microdifferential operator to the generator $1/\Phi$. Finally, one says that a $\mathcal{D}$-linear morphism is an m-f-c isomorphism (short hand for isomorphism modulo flat connections) if its kernel and cokernel are locally free $\mathcal{O}$-modules of finite rank.

PROPOSITION 2.4. *With the above notations and hypotheses* ★, *let $\mathcal{H}$ be a holomorphic vector bundle on X, and let $\mathcal{N}$ be a $\mathcal{D}_\Xi$-module with simple characteristics along $\dot{W}$. Let $\kappa \in H^0 \mathrm{Sol}((\mathcal{D}\mathcal{H}^*)^\vee \overset{D}{\boxtimes} \mathcal{N}, \mathcal{K})$ be a non-degenerate section of $\mathcal{B}_S$. Then,*

$$H^0 \alpha(\kappa) \colon \mathcal{N} \to H^0(\mathcal{D}\mathcal{H}^* \overset{D}{\circ} \mathcal{B}_S)$$

is an m-f-c isomorphism.

Applying (2.3) with $G = \mathbb{C}_\xi$, $\xi \in \Xi$, and $\natural = \omega$, we get the germ formula

$$(2.8) \qquad (\mathcal{D}\mathcal{H}^* \overset{D}{\circ} \mathcal{B}_S)_\xi \simeq \mathrm{R}\Gamma(\widehat{\xi}; \mathcal{H}|_{\widehat{\xi}})[\dim S - \dim \Xi].$$

This, together with the similar formula for $(\mathcal{D}\mathcal{H}^*)^\vee$, provide a useful test to check both if $\alpha(\kappa) = H^0\alpha(\kappa)$ and if $\alpha(\kappa)$ is an actual isomorphism, not only modulo flat connections.

One says that a $\mathcal{D}_{\Xi}$-module $\mathcal{N}$ has regular singularities along $\dot{W}$ if it locally admits a presentation

$$\mathcal{S}^{N_1} \to \mathcal{S}^{N_0} \to \mathcal{N} \to 0,$$

where $\mathcal{S}$ has simple characteristics along $\dot{W}$. Assume

★ f has connected and simply connected fibers.

THEOREM 2.5. *With the above notations and hypotheses ★, the functor* $\mathcal{M} \mapsto H^0(\mathcal{M} \overset{D}{\circ} \mathcal{B}_S)$ *induces an equivalence of categories between coherent* $\mathcal{D}_X$-*modules, modulo flat connections, and* $\mathcal{D}_{\Xi}$-*modules with regular singularities along* $\dot{W}$, *modulo flat connections. Moreover, this equivalence interchanges modules with simple characteristics along* $\dot{T}^*X$ *with modules with simple characteristics along* $\dot{W}$. *Finally, if* $\dim X \geq 3$, *any* $\mathcal{D}_X$-*module with simple characteristics along* $\dot{T}^*X$ *is m-f-c isomorphic to a module of the form* $\mathcal{D}\mathcal{H}^*$, *for some line bundle* $\mathcal{H}$ *on* X.

Notes. §2.1 The idea of constructing generalized functions starting with holomorphic functions is at the heart of Sato's theory of hyperfunctions. When making operations on hyperfunctions one is then led to consider complexes of the form $\mathcal{C}^{\pm\omega}(F)$. This theory is developed in [KSa90], which is also a very good reference on sheaf theory in derived category.

By taking into account growth conditions, distributions can also be obtained from the sheaf of holomorphic functions. This is done using the functor $t$hom of tempered cohomology introduced in [Kas84]. The dual construction of the formal cohomology functor $\overset{w}{\otimes}$ is performed in [KSa96], where one also finds a systematic treatment of operations on sol$(\mathcal{M}, \mathcal{C}^{\pm\infty}(F))$.

§2.2 Kashiwara's Master Thesis, recently translated in [Kas70], is still a very good reference on the analytic theory of $\mathcal{D}$-modules. See also [SKK73], [Bjo93] and [Scn94]. A proof of the Riemann-Hilbert correspondence is obtained in [Kas84] by showing that the functor $t$hom$(\cdot, \mathcal{O}_X)$ is a quasi-inverse to the solution functor sol$(\cdot, \mathcal{O}_X)$.

§2.3 The language of correspondences is very classical. In the category of sheaves, it can be found for example in [KSa90]. The formalism of sol$(\mathcal{M}, \mathcal{C}^{\pm\omega}(F))$ was introduced in [SS94]. The idea of using this framework to investigate integral transforms is from [DS96a].

The starting point to address problems in integral geometry is the adjunction formula in Theorem 2.1. For $\natural = \pm\omega$ this was obtained in [DS96a], [DS96b] using results as the Cauchy-Kovalevskaya-Kashiwara theorem and Schneiders' relative duality theorem. The case $\natural = \pm\infty$ is from [KSa96]. Note also that [KSm94] independently announced a similar result in an equivariant framework, at the time of our announcement [A. D'Agnolo and P. Schapira, C. R. Acad. Sci. Paris Sér. I Math. **319**, no. 5 (1994), 461–466; Ibid. no. 6, 595–598].

The kernel lemma 2.2 is from [DS96b] (see also [Gnc97] and [Tan98]).

We refer to the appendix of [Dag98] for a discussion on how to obtain the distribution kernel k as boundary value of the meromorphic kernel κ.

§2.5 The microlocal geometry attached to double fibrations was first considered in [GS79]. Proposition 2.3 is from [DS96a]. Its proof relies on a result asserting that

the functor $\cdot \overset{D}{\circ} \mathcal{B}_S \simeq Dg_! Df^*(\cdot)$ commutes to microlocalization. The corresponding result for inverse images is due to [SKK73], and the one for direct images is due to [SS94].

Proposition 2.4 is from [DS96b], [DS98]. Its proof is based on the quantization of contact transformations by [SKK73], where the notion of non degenerate section is found (see also [Scp85] for an exposition of the theory of microdifferential operators). The equivalence result in Theorem 2.5 was obtained in [DS96a], [DS98]. The case $X = \mathbb{P}$, $\Xi = \mathbb{P}^*$ had already been considered by Brylinski [Bry86] for regular holonomlic modules.

3. Back to the Radon transform

Using the framework of sheaves and $\mathcal{D}$-modules, we will give here a proof of the statements in Section 1, along with some generalizations.

According to (1.4), we denote by $\mathbb{P}$ the projective space of vector lines in $\mathbb{V} \simeq \mathbb{C}^{n+1}$, and by $\mathbb{G}$ the Grassmannian of $(p+1)$-dimensional subspaces of $\mathbb{V}$. Let $\mathbb{F} \subset \mathbb{P} \times \mathbb{G}$ be the incidence relation of pairs (z, ζ) with $z \in \widehat{\zeta}$. Recall that $\dim \mathbb{P} = n$, $\dim \mathbb{G} = (p+1)(n-p)$, and $\mathbb{F}$ is a smooth submanifold of $\mathbb{P} \times \mathbb{G}$ of dimension $n + p(n-p)$. By definition, $\mathbb{F}$ is the graph

$$(3.1) \qquad \mathbb{P} \underset{f}{\leftarrow} \mathbb{F} \underset{g}{\rightarrow} \mathbb{G}$$

of the correspondence $\mathbb{G} \ni \zeta \multimap \widehat{\zeta} \subset \mathbb{P}$ attached to the complex Radon transform. Let us describe the associated microlocal correspondence

$$(3.2) \qquad T^*\mathbb{P} \underset{p_1}{\longleftarrow} T^*_{\mathbb{F}}(\mathbb{P} \times \mathbb{G}) \underset{p_2^a}{\longrightarrow} T^*\mathbb{G}.$$

Denote by $\langle \zeta \rangle \subset \mathbb{V}$ the $(p+1)$-vector subspace attached to $\zeta \in \mathbb{G}$. It is a nice exercise in classical geometry[*] to recover the formula $(T\mathbb{G})_\zeta = \mathrm{Hom}\,(\langle \zeta \rangle, \mathbb{V}/\langle \zeta \rangle)$, where this Hom is simply the set of linear morphisms of vector spaces. Similarly, one has

$$
\begin{aligned}
T^*\mathbb{P} &= \{(z; \alpha) \colon \alpha \in \mathrm{Hom}\,(\mathbb{V}/\langle z \rangle, \langle z \rangle)\}, \\
T^*\mathbb{G} &= \{(\zeta; \beta) \colon \beta \in \mathrm{Hom}\,(\mathbb{V}/\langle \zeta \rangle, \langle \zeta \rangle)\}, \\
T^*_{\mathbb{F}}(\mathbb{P} \times \mathbb{G}) &= \{(z, \zeta; \gamma) \colon \gamma \in \mathrm{Hom}\,(\mathbb{V}/\langle \zeta \rangle, \langle z \rangle)\}.
\end{aligned}
$$

The maps (3.2) are then given by

$$p_1(z, \zeta; \gamma) = (z; \gamma \circ q), \qquad p_2^a(z, \zeta; \gamma) = (\zeta; j \circ \gamma),$$

where $q \colon \mathbb{V}/\langle z \rangle \twoheadrightarrow \mathbb{V}/\langle \zeta \rangle$ and $j \colon \langle z \rangle \hookrightarrow \langle \zeta \rangle$ are induced by the inclusion $\langle z \rangle \subset \langle \zeta \rangle$.

REMARK. One should be careful that, with the notations (1.5),

$$\langle \zeta \rangle = (\mathbb{C} \cdot \zeta_1 + \cdots + \mathbb{C} \cdot \zeta_{n-p})^\perp \subset \mathbb{V}.$$

This is due to the fact that (1.5) uses the identification $\zeta \mapsto \zeta^\perp$ of $\mathbb{G}$ with the Grassmannian of $(n-p)$-dimensional subspaces of $\mathbb{V}^*$.

It is now easy to check that all of the hypotheses ★ in the previous section are satisfied for the choice

$$\blacktriangle \quad X = \mathbb{P}, \quad \Xi = \mathbb{G}, \quad S = \mathbb{F}, \quad \mathcal{K} = \mathcal{B}_{\mathbb{F}}, \quad K = \mathbb{C}_{\mathbb{F}}[p-n].$$

---

[*]see for example [J. Harris, *Algebraic geometry. A first course.* Graduate Texts in Mathematics, 133. Springer-Verlag, New York, 1995, MR **97e:**14001]

Note also that $W = p_2^a(p_1^{-1}(T^*\mathbb{P}))$ is described by

$\blacktriangle$ $W = \{(\zeta; \beta) \in T^*\mathbb{G} : \beta \colon \mathbb{V}/\langle\zeta\rangle \to \langle\zeta\rangle$ has rank at most one$\}$.

By Proposition 2.3 (i), W is the *a priori* estimate for the characteristic variety of any $\mathcal{D}_\mathbb{G}$-module that can possibly arise from the Radon transform. Not too surprisingly, this is the characteristic variety of the Maxwell-John system $\square$.

Our aim here is to show that all of the examples from Section 1 are particular cases of Theorem 2.1, for different choices of $\alpha\colon \mathcal{N} \to \mathcal{M} \overset{D}{\circ} \mathcal{B}_\mathbb{F}$ and $\beta\colon F \to \mathbb{C}_\mathbb{F} \circ G_{[p-n]}$. Even better, for those examples α is always the same.

3.1. A Radon $\mathcal{D}$-module. The real affine Radon transform in Theorem 1.1 deals with the space $\mathcal{S}(\mathsf{A})$, where no differential equations appear. Theorems 1.2 and 1.3 are concerned with function spaces attached to the line bundle $\mathcal{O}_\mathbb{P}(-p-1)$. Again, this is something which is locally trivial. It is then natural to choose $\mathcal{M} = \mathcal{D}_{\mathbb{P}(p+1)}$, where we set $\mathcal{D}_{\mathbb{P}(m)} = \mathcal{D}_\mathbb{P} \otimes_{\mathcal{O}_\mathbb{P}} \mathcal{O}_\mathbb{P}(m)$ for $m \in \mathbb{Z}$.

Concerning $\mathcal{N}$, if $p < n-1$ this should be the $\mathcal{D}_\mathbb{G}$-module $\mathcal{D}_\mathbb{G}^\square$ represented by the Maxwell-John system (1.7)

$$\mathcal{D}\mathcal{H}^* \underset{\square}{\longrightarrow} \mathcal{D}_{\mathbb{G}}(1) \to \mathcal{D}_\mathbb{G}^\square \to 0.$$

If $p = n-1$, then $\mathbb{G} = \mathbb{P}^*$ is a dual projective space and we should take $\mathcal{N} = \mathcal{D}_{\mathbb{P}^*}(1)$. Since in this case $\square = 0$, we may still write $\mathcal{D}_{\mathbb{P}^*}(1) = \mathcal{D}_{\mathbb{P}^*}^\square$.

Finally, the choice of κ is imposed by (1.6). Summarizing, we consider

$\blacktriangle$ $\mathcal{M} = \mathcal{D}_{\mathbb{P}(p+1)}, \quad \mathcal{N} = \mathcal{D}_\mathbb{G}^\square, \quad (2\pi i)^{n-p}\kappa(z,\zeta) = \dfrac{\omega(z)}{\langle z,\zeta_1\rangle \cdots \langle z,\zeta_{n-p}\rangle}.$

To explain the meaning of κ, note that the Čech covering $\{\langle z,\zeta_i\rangle \neq 0\}_{i=1,\dots,n-p}$ of $(\mathbb{P} \times \mathbb{G}) \setminus \mathbb{F}$ allows us to locally consider κ as a cohomology class in $H_{[\mathbb{F}]}^{n-p}\mathcal{O}_{\mathbb{P}\times\mathbb{G}}$. Globally, $\kappa(z,\zeta)$ is $(p+1)$-homogeneous in z, and (-1)-homogeneous in ζ, for the action of $\mathrm{GL}(1,\mathbb{C})$ and $\mathrm{GL}(n-p,\mathbb{C})$, respectively. This is written as

$$\kappa \in H^0 \operatorname{Sol}(\mathcal{D}_{\mathbb{P}(p+1)}{}^\vee \overset{D}{\boxtimes} \mathcal{D}_{\mathbb{G}}(1), \mathcal{B}_\mathbb{F}).$$

Moreover, it is easy to check that $\kappa(z,\zeta)$ is a solution of $\square$, acting on the ζ variable. Hence

$$\kappa \in H^0 \operatorname{Sol}(\mathcal{D}_{\mathbb{P}(p+1)}{}^\vee \overset{D}{\boxtimes} \mathcal{D}_\mathbb{G}^\square, \mathcal{B}_\mathbb{F}).$$

It is now a local problem on $\dot{T}_\mathbb{F}^*(\mathbb{P} \times \mathbb{G}) \overset{\sim}{\longrightarrow} \dot{W}$ to verify that $\mathcal{D}_\mathbb{G}^\square$ is simple along $\dot{W}$, and that κ is non-degenerate.

THEOREM 3.1. *The above choice of κ induces an isomorphism*

$$\alpha(\kappa)\colon \mathcal{D}_\mathbb{G}^\square \overset{\sim}{\longrightarrow} \mathcal{D}_{\mathbb{P}(p+1)} \overset{D}{\circ} \mathcal{B}_\mathbb{F}.$$

REMARK. The above discussion was facilitated by the fact that the choice of $\mathcal{M}$, $\mathcal{N}$ and κ was forced by the statements we wanted to recover. In general, one either has good candidates for $\mathcal{N}$ and $\mathcal{K}$, and may then proceed as above, or one has to actually compute the transform $\mathcal{M} \overset{D}{\circ} \mathcal{K}$. This last problem is generally difficult. Note, however, that many of the examples that arise in practice are endowed with the action of a group, including the Radon transform which is equivariant for the action of $\mathrm{GL}(\mathbb{V})$. Taking this into account can greatly simplifies matters, by narrowing down the possible outcome of the transform, and by allowing one to use the computational techniques of representation theory.

3.2. Radon adjunction formula. For an $\mathcal{O}_{\mathbb{P}}$-module $\mathcal{F}$, set $\mathcal{F}(m) = \mathcal{F} \otimes_{\mathcal{O}_{\mathbb{P}}} \mathcal{O}_{\mathbb{P}}(m)$. Combining Theorems 2.1 and 3.1 we get

THEOREM 3.2. *Let $\beta \colon F \to \mathbb{C}_{\mathbb{F}} \circ G$ be a $\mathbb{C}$-linear morphism inducing an isomorphism*

$$(3.3) \qquad H^p(\mathbb{P}; \mathcal{C}^{\natural}(F)(-p-1)) \xrightarrow{\sim} H^p(\mathbb{P}; \mathcal{C}^{\natural}(\mathbb{C}_{\mathbb{F}} \circ G)(-p-1)).$$

Then, one has an isomorphism

$$\beta \cdot \alpha(\kappa) \colon H^p(\mathbb{P}; \mathcal{C}^{\natural}(F)(-p-1)) \xrightarrow{\sim} H^0 \operatorname{Sol}(\mathcal{D}_{\mathbb{G}}^{\square}, \mathcal{C}^{\natural}(G)).$$

Requiring that β itself be an isomorphism is of course a sufficient condition for (3.3) to hold, but it is not necessary. For example, since $\mathrm{R}\Gamma(\mathbb{P}; \mathcal{O}_{\mathbb{P}}(-p-1)) = 0$, one has that $\mathrm{R}\Gamma(\mathbb{P}; \mathcal{C}^{\natural}(N)(-p-1)) = 0$ if N is a complex of finite rank constant sheaves on $\mathbb{P}$. In other words, the morphism $\mathrm{R}\Gamma(\mathbb{P}; \mathcal{C}^{\natural}(\beta)(-p-1))$ is an isomorphism if β is an isomorphism in the localization of $\mathbf{D}_{\mathbb{R}-c}^{\mathrm{b}}(\mathbb{C}_{\mathbb{P}})$ by the null systems of objects like N. We denote this localization by $\mathbf{D}_{\mathbb{R}-c}^{\mathrm{b}}(\mathbb{C}_{\mathbb{P}}; \dot{T}^*\mathbb{P})$.

We are now in a position to deduce all of the results in Section 1 from Theorem 3.2, for different choices of $\beta \colon F \to \mathbb{C}_{\mathbb{F}} \circ G$.

3.3. Complex projective case. We use the same notations as in Section 1.2. Recall that

$$\mathcal{C}^{-\omega}(\mathbb{C}'_{\widehat{U}}) = \mathrm{R}\Gamma_{\widehat{U}}(\mathcal{O}_{\mathbb{P}}), \quad \mathcal{C}^{-\omega}(\mathbb{C}'_U) = \mathrm{R}\Gamma_U(\mathcal{O}_{\mathbb{G}}).$$

To get Theorem 1.2 as a corollary of Theorem 3.2, one should then take

▲ $F = \mathbb{C}'_{\widehat{U}}, \quad G = \mathbb{C}'_U, \quad \natural = -\omega$.

Note that, by duality, the datum of a morphism

$$\beta \colon F \to \mathbb{C}_{\mathbb{F}} \circ G$$

is equivalent to the datum of

$$\beta' \colon F' \leftarrow (\mathbb{C}_{\mathbb{F}} \circ G)' \simeq \mathbb{C}_{\mathbb{F}} \circ G'[2p(n-p)].$$

According to (2.5), for $z \in \mathbb{P}$ one has

$$H^j(\mathbb{C}_{\mathbb{F}} \circ \mathbb{C}_U)_z \simeq H_c^j(\widehat{z}; \mathbb{C}_{\widehat{z} \cap U})$$
$$\simeq H^{2p(n-p)-j}(\widehat{z}; \mathbb{C}_{\widehat{z} \cap U}),$$

where the last isomorphism is Poincaré duality. The set $\widehat{z} \cap U$ is non-empty if and only if $z \in \widehat{U}$. If $\widehat{z} \cap U$ is connected for any $z \in \widehat{U}$, we get a morphism

$$\beta' \colon \mathbb{C}_{\widehat{U}} \leftarrow \mathbb{C}_{\mathbb{F}} \circ \mathbb{C}_U[2p(n-p)].$$

Since U is elementary, the truncation $\tau^{\geq -p}(\beta')$ of β' in degree greater or equal to $-p$ is an isomorphism. From this fact one deduces (3.3), and the statement follows.

REMARK. The technical point of passing from β to β' is inessential and could have been avoided. It is solely due to our definition of $\mathcal{C}^{-\omega}(F)$, which incorporates F' to be best suited for the real case.

Of course, we could also consider the cases $\natural = \omega, \pm\infty$. For example, take

▲ $p = n-1, \quad F = \mathbb{C}_{\widehat{U}}, \quad G = \mathbb{C}_U[2(n-1)], \quad \natural = \omega$.

If U is a bounded neighborhood of the origin in an affine chart $\mathbb{A}^* \subset \mathbb{P}^*$, then $U^\sharp = \mathbb{P} \setminus \widehat{U}$ is a compact subset in an affine chart $\mathbb{A} \subset \mathbb{P}$. We thus recover Martineau's isomorphism

$$\Gamma(U^\sharp; \mathcal{O}_\mathbb{A}) \simeq \mathcal{O}'_{\mathbb{A}^*}(U),$$

where $\mathcal{O}'_{\mathbb{A}^*}(U) = H^n_c(U; \mathcal{O}_{\mathbb{A}^*}) = H^n_c(U; \mathcal{O}_{\mathbb{P}^*}(-1))$ is the space of analytic functionals in U.

3.4. Real conformal case. We use the same notations as in Section 1.3. By definition, we have

$$\mathrm{R}\Gamma(\mathbb{P}; \mathcal{C}^\omega(\mathbb{C}_\mathsf{Q})) \simeq \mathrm{R}\Gamma(\mathsf{Q}; \mathcal{O}_\mathbb{P}), \quad \mathrm{R}\Gamma(\mathbb{P}; \mathcal{C}^{-\omega}(\mathbb{C}_\mathsf{Q})) \simeq \mathrm{R}\Gamma_\mathsf{Q}(\mathbb{P}; \mathcal{O}_\mathbb{P})[1],$$

where in the second isomorphism we used the identification $\mathbb{C}'_\mathsf{Q} \simeq \mathbb{C}_\mathsf{Q}[1]$, due to the fact that Q is a smooth hypersurface splitting $\mathbb{P}$ in two connected components. Since M is totally real in $\mathbb{G}$, we also have

$$\mathcal{C}^\omega(\mathbb{C}_\mathsf{M}) = \mathcal{C}^\omega_\mathsf{M}, \quad \mathcal{C}^{-\omega}(\mathbb{C}_\mathsf{M}) = \mathcal{C}^{-\omega}_\mathsf{M}.$$

Take

▲ $F = \mathbb{C}_\mathsf{Q}, \quad G = \mathbb{C}_\mathsf{M}, \quad \natural = \pm\omega,$

and let β be the morphism induced by the equality $\widehat{\mathsf{M}} = \mathsf{Q}$. Then, the choice $\natural = \omega$ allows to get the isomorphism in the first line of Theorem 1.3, while $\natural = -\omega$ gives the one in the second line.

3.5. Real projective case. Let us discuss this case in somewhat more detail. Consider

(3.4) $\begin{cases} \mathsf{V} \simeq \mathbb{R}^{n+1} & \text{a real vector space with } \mathsf{V} \otimes_\mathbb{R} \mathbb{C} = \mathbb{V}, \\ \mathsf{P} & \text{the projective space of vector lines in } \mathsf{V}, \\ \mathsf{G} & \text{the Grassmannian of projective } p\text{-planes in } \mathsf{P}. \end{cases}$

Since $\pi_1(\mathsf{P}) = \mathbb{Z}/2\mathbb{Z}$ (let us assume here that $n > 1$), there are essentially two locally constant sheaves of rank one on P. For $\varepsilon \in \mathbb{Z}/2\mathbb{Z}$, we denote them by $\mathbb{C}_\mathsf{P}^{(\varepsilon)}$, asking that $\mathbb{C}_\mathsf{P}^{(0)}$ be the constant sheaf $\mathbb{C}_\mathsf{P}$. One then easily checks that

$$\mathcal{C}^\infty(\mathbb{C}_\mathsf{P}^{(-p-1)})(-p-1) \simeq \mathcal{C}_\mathsf{P}^\infty\{-p-1\},$$

where the term on the right hand side denotes the C^∞ line bundle on P whose sections φ satisfy the homogeneity condition

$$\varphi(\lambda x) = |\lambda|^{-p-1}\varphi(x) \quad \forall \lambda \in \mathrm{GL}(1, \mathbb{R}).$$

Since $\pi_1(\mathsf{G}) = \mathbb{Z}/2\mathbb{Z}$, in exactly the same way we have

$$\mathcal{C}_\mathsf{G}^\infty\{-1\} = \mathcal{C}^\infty(\mathbb{C}_\mathsf{G}^{(-1)})(-1).$$

(Note that if $q\colon \mathsf{G}^+ \to \mathsf{G}$ denotes the $2:1$ projection from the Grassmannian of oriented planes to G, one has $q_*(\mathbb{C}_{\mathsf{G}^+}) = \mathbb{C}_\mathsf{G} \oplus \mathbb{C}_\mathsf{G}^{(1)}$.)

Let us set

▲ $F = \mathbb{C}_\mathsf{P}^{(-p-1)}, \quad G = \mathbb{C}_\mathsf{G}^{(-1)}, \quad \natural = \infty.$

Denote by $\mathsf{G}_{p,n}$ the Grassmannian of p-dimensional subspaces of $\mathbb{R}^n$. For $z \in \mathbb{P}$ one has

$$\widehat{z} \cap \mathsf{G} = \{\xi \in \mathsf{G}\colon \mathbb{C} \otimes_\mathbb{R} \langle\xi\rangle \supset \langle z\rangle\} \simeq \begin{cases} \mathsf{G}_{p,n}, & \text{for } z \in \mathsf{P}, \\ \mathsf{G}_{p-1,n-1}, & \text{for } z \in \mathbb{P} \setminus \mathsf{P}. \end{cases}$$

By a computation like (2.5), this implies

$$H^j(\mathbb{C}_{\mathbb{F}} \circ \mathbb{C}_{\mathsf{G}}^{(-1)})_z \simeq \begin{cases} H^j(\mathsf{G}_{p,n}; \mathbb{C}_{\mathsf{G}_{p,n}}^{(-1)}), & \text{for } z \in \mathsf{P}, \\ H^j(\mathsf{G}_{p-1,n-1}; \mathbb{C}_{\mathsf{G}_{p-1,n-1}}^{(-1)}), & \text{for } z \in \mathbb{P} \setminus \mathsf{P}. \end{cases}$$

Looking at a table of Betti numbers for oriented and non-oriented real Grassmannians, one deduces that

$$\tau^{\leq p}(\mathbb{C}_{\mathbb{F}} \circ \mathbb{C}_{\mathsf{G}}^{(-1)}) \simeq \begin{cases} \mathbb{C}_{\mathsf{P}}^{(-p-1)}, & \text{for } p \text{ even}, \\ \mathbb{C}_{\mathbb{P} \setminus \mathsf{P}}, & \text{for } p \text{ odd}. \end{cases}$$

Denoting by

$$\beta \colon \mathbb{C}_{\mathsf{P}}^{(-p-1)} \to \mathbb{C}_{\mathbb{F}} \circ \mathbb{C}_{\mathsf{G}}^{(-1)}$$

the natural morphism, the above arguments imply that $\tau^{\leq p}(\beta)$ is an isomorphism in $\mathbf{D}_{\mathbb{R}-c}^{b}(\mathbb{C}_{\mathbb{P}}; \dot{T}^*\mathbb{P})$. Applying Theorem 3.2, we get a compactified version of Theorem 1.1.

THEOREM 3.3. *The real projective Radon transform*

$$R_{\mathsf{P}} \colon \Gamma(\mathsf{P}; \mathcal{C}_{\mathsf{P}}^{\infty}\{-p-1\}) \to \Gamma(\mathsf{G}; \mathcal{C}_{\mathsf{G}}^{\infty}\{-1\})$$

$$\varphi(x) \mapsto \psi(\xi) = \int \varphi(x) \delta(\langle x, \xi_1 \rangle) \cdots \delta(\langle x, \xi_{n-p} \rangle) \omega(x)$$

induces an isomorphism

$$R_{\mathsf{P}} \colon \Gamma(\mathsf{P}; \mathcal{C}_{\mathsf{P}}^{\infty}\{-p-1\}) \xrightarrow{\sim} \Gamma(\mathsf{G}; \ker(\Box, \mathcal{C}_{\mathsf{G}}^{\infty})).$$

Here, we used the identification $R_{\mathsf{P}} = \beta \cdot \alpha(\kappa)$, which follows from the fact that

$$k(x, \xi) = \delta(\langle x, \xi_1 \rangle) \cdots \delta(\langle x, \xi_{n-p} \rangle) \omega(x)$$

is the boundary value of $\kappa(z, \zeta)$.

3.6. Real affine case. The affine case (1.1) sits in the projective case (3.4) by considering

(3.5) $$\begin{cases} \mathsf{A} = \mathsf{P} \setminus \mathsf{H} & \text{for a hyperplane } \mathsf{H} \subset \mathsf{P}, \\ \mathsf{G}_{\mathsf{A}} \subset \mathsf{G} & \text{the set of } \xi \in \mathsf{G} \text{ with } \widehat{\xi} \subset \mathsf{A}. \end{cases}$$

The space $\mathcal{S}(\mathsf{A})$ is then identified to the space of C^∞-functions globally defined in P, which vanish up to infinite order on H. Since twisting does not matter in affine charts, we have

$$\mathcal{S}(\mathsf{A}) \simeq \Gamma(\mathbb{P}; \mathcal{C}^\infty(\mathbb{C}_{\mathsf{A}})(-p-1)).$$

To recover Theorem 1.1, one then has to consider

▲ $F = \mathbb{C}_{\mathsf{A}} \simeq \mathbb{C}_{\mathsf{A}}^{(-p-1)}, \quad G = \mathbb{C}_{\mathsf{G}_{\mathsf{A}}}^{(-1)}, \quad \natural = \infty,$

where $\mathbb{C}_{\mathsf{G}_{\mathsf{A}}}^{(-1)} = \mathbb{C}_{\mathsf{G}_{\mathsf{A}}} \otimes \mathbb{C}_{\mathsf{G}}^{(-1)}$. Let us detail the more interesting case $p = n - 1$, where the Cavalieri condition appears.

We are now considering the situation

(3.6) $$\begin{cases} \mathsf{G} = \mathsf{P}^* & \text{a real projective space in } \mathbb{G} = \mathbb{P}^*, \\ \mathsf{G}_{\mathsf{A}} = \mathsf{P}_\circ^* & \text{where } \mathsf{P}_\circ^* = \mathsf{P}^* \setminus \{h\} \text{ for } h \in \mathsf{P}^* \text{ with } \widehat{h} = \mathsf{H}. \end{cases}$$

Since the Radon hyperplane transform is symmetrical, we may apply Theorem 3.2 interchanging the roles of $\mathbb{P}$ and $\mathbb{P}^*$. We get

(3.7) $$R_{\mathsf{A}}(\mathcal{S}(\mathsf{A})) \simeq H^n(\mathbb{P}^*; \mathcal{C}^\infty(\mathbb{C}_{\mathsf{A}} \circ \mathbb{C}_{\mathbb{F}})(-1)).$$

The embedding $\mathsf{H} \subset \mathsf{P}$ induces a projection $\mathsf{P}_\circ^* \to \mathsf{H}^*$, and its complexification

$$q \colon \mathbb{P}_\circ^* \to \mathbb{H}^*,$$

where H^* is the dual projective space to H. Note that the fibers of q are the complex projective lines through h, with the point h removed. Consider

(3.8) $$\tilde{\mathsf{P}}_\circ^* = q^{-1}(\mathsf{H}^*), \quad \tilde{\mathsf{P}}^* = \tilde{\mathsf{P}}_\circ^* \cup \{h\}.$$

The set $\tilde{\mathsf{P}}^*$ has its only singularity at h. An explicit computation gives the distinguished triangle in $\mathbf{D}_{\mathbb{R}-\mathrm{c}}^{\mathrm{b}}(\mathbb{C}_\mathbb{P}; \dot{T}^*\mathbb{P})$

$$\mathbb{C}_\mathsf{A} \circ \mathbb{C}_\mathbb{F} \to \mathbb{C}_{\mathsf{P}_\circ^*}^{(-1)}[-n] \to \mathbb{C}_{\tilde{\mathsf{P}}^*}[1-n] \xrightarrow[+1]{} \; .$$

Using the notation $\mathcal{S}_{(-1)}(\tilde{\mathsf{P}}^*) = \mathrm{R}\Gamma(\mathbb{P}^*; \mathcal{C}^\infty(\mathbb{C}_{\tilde{\mathsf{P}}^*})(-1))$, we may then write (3.7) as

$$R_\mathsf{A}(\mathcal{S}(\mathsf{A})) = \ker\left(\mathcal{S}_{\{-1\}}(\mathsf{P}_\circ^*) \xrightarrow[c]{} H^1\mathcal{S}_{(-1)}(\tilde{\mathsf{P}}^*)\right),$$

and we are left to describe the morphism c.

The decomposition (3.8) gives an identification

$$H^1\mathcal{S}_{(-1)}(\tilde{\mathsf{P}}^*) = \mathrm{coker}\left(\mathcal{O}_{\mathbb{P}^*}(-1)\widehat{|}_h \xrightarrow[j]{} H^1\mathcal{S}_{(-1)}(\tilde{\mathsf{P}}_\circ^*)\right).$$

Here $\mathcal{O}_{\mathbb{P}^*}(-1)\widehat{|}_h = H^1\mathcal{S}_{(-1)}(\{h\})$ is the formal restriction of $\mathcal{O}_{\mathbb{P}^*}(-1)$ to h, whose sections are formal Taylor series

$$\mathcal{O}_{\mathbb{P}^*}(-1)\widehat{|}_h \simeq \prod_{m \geq 0} \Gamma(\mathbb{H}^*; \mathcal{O}_{\mathbb{H}^*}(m)).$$

Summarizing, we have

$$\mathcal{S}_{\{-1\}}(\mathsf{P}_\circ^*) \xrightarrow[\tilde{c}]{} \prod_{m \geq 0} \Gamma(\mathsf{H}^*; \mathcal{C}_{\mathsf{H}^*}^\infty(m)) \xleftarrow{\sim} H^1\mathcal{S}_{(-1)}(\tilde{\mathsf{P}}_\circ^*) \xleftarrow[j]{} \prod_{m \geq 0} \Gamma(\mathbb{H}^*; \mathcal{O}_{\mathbb{H}^*}(m)),$$

where $\tilde{c}$ and the middle isomorphism are obtained by integration along the fibers of q. Using the above identifications, we finally get

$$R_\mathsf{A}(\mathcal{S}(\mathsf{A})) = \{\psi \in \mathcal{S}_{\{-1\}}(\mathsf{P}_\circ^*) \colon \tilde{c}(\psi)_m = j(\Psi_m) \text{ for some } \Psi_m \in \Gamma(\mathbb{H}^*; \mathcal{O}_{\mathbb{H}^*}(m))\}.$$

This implies Theorem 1.1 (ii), by the following considerations. Take a system of homogeneous coordinates $[\xi] = [\xi_0, \xi']$ in P^* such that $h = [1, 0, \ldots, 0]$, H^* is given by the equation $\xi_0 = 0$, $[\xi']$ are homogeneous coordinates in H^*, and $q([\xi]) = [\xi']$. Then

$$\tilde{c}(\psi)_m(\xi') = \int_{-\infty}^{+\infty} \psi(\sigma\xi_0 + \xi')\sigma^m \, d\sigma.$$

Moreover, $\Gamma(\mathbb{H}^*; \mathcal{O}_{\mathbb{H}^*}(m))$ is precisely the space of homogeneous polynomials of degree m in ξ', considered as global sections of $\mathcal{C}_{\mathsf{H}^*}^\infty(m)$.

Notes. §3.1 Theorem 3.1 was obtained in [DS96b] for $p = n - 1$, where one may also find another proof based on the Cauchy-Fantappiè formula. The case $p < n - 1$ is from [DM99a]. In these two papers one also finds a discussion of the case $\mathcal{M} = \mathcal{D}_\mathbb{P}(m)$, where $\mathcal{N} = \mathcal{M} \overset{D}{\circ} \mathcal{B}_\mathbb{F}$ is associated to the higher dimensional analog of the zero-rest-mass field equations. Note that Tanisaki [Tan98] has an alternative representation theoretical proof for the case $m = -p - 1$, which extends to other flag manifolds. This should also be related to the work of Oshima [Osh96].

§§**3.3–3.5** The proofs in these sections are from [DS96b] for the case $p = n - 1$, and from [DS96a] for the case $p = 1$, $n = 3$. The general case $p < n - 1$ was later obtained in [DM99b].

In a classical framework, Theorem 3.3 can be found in [GGG82] or [Hel84].

The idea of investigating the real Radon transform through the complex one, which is intrinsic in our approach, also appears in [Gin98]. For the case $p = 1$, $n = 3$, another close approach can be found in [Eas97].

§**3.6** The geometric approach to the Cavalieri condition is from [Dag98].

The case $p < n - 1$ is treated in [DM99b].

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